6.2: SOLVING WITH LAPLACE TRANSFORMS

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \qquad u = e^{-st} \quad dv = f'(t)dt$$
$$du = -se^{-st}dt \quad v = f(t)$$

$$= \lim_{A\to\infty}\left[e^{-st} f(t)\Big|_0^A - \int_0^A -se^{-st} f(t)\,dt \right]$$

$$= \lim_{A\to\infty}\left[e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)\,dt \right]$$

$$= \quad - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \boxed{\mathcal{L}\{f'(t)\} = - f(0) + s\,\mathcal{L}\{f(t)\}}$$

Similarly

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)}$$

$$\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

EX $\quad y'' - y' - 2y = 0, \quad y(0) = 1, \; y'(0) = 0$

I $\quad \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$

II $\quad \left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - \left(s\mathcal{L}\{y\} - y(0)\right) - 2\mathcal{L}\{y\} = 0$

$$(s^2 - s - 2)\mathcal{L}\{y\} - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} + \underbrace{y(0)}_{1} = 0$$

$$(s^2 - s - 2)\mathcal{L}\{y\} = s - 1$$

$$\boxed{\mathcal{L}\{y\} = \frac{s-1}{s^2 - s - 2}}$$

III PARTIAL FRACTIONS

PARTIAL FRACTION HANDOUT!

$$\frac{s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

DISTINCT LINEAR TERMS

$$s-1 = A(s+1) + B(s-2)$$

WHATEVER YOU WANT

OPTION 1

$s-1 = (A+B)s + (A-2B)$

$A+B = 1$ $\quad A = 1/3$

$A-2B = -1$ $\quad B = 2/3$

OPTION 2

$s=2 \Rightarrow 1 = A(3)$

$s=-1 \Rightarrow -2 = B(-3)$

$A = 1/3$

$B = 2/3$

OPTION 3 COVER UP METHOD

$A = \frac{2-1}{2+1} = 1/3$

$B = \frac{-1-1}{-1-2} = \frac{-2}{-3} = 2/3$

FIND A METHOD THAT WORKS FOR YOU!

$$\Rightarrow \mathcal{L}\{y\} = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

$\boxed{\text{IV}}$ INVERSE!

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

SUMMARY

$\boxed{\text{I}}$ LAPLACE TRANSFORM BOTH SIDES!

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

$\boxed{\text{II}}$ REARRANGE & ISOLATE $\mathcal{L}\{y\}$ & PUT IN INITIAL COND.

$\boxed{\text{III}}$ PARTIAL FRACTIONS

$\boxed{\text{IV}}$ INVERSE LAPLACE (USE TABLE!)

Ex) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

$$\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) - 4\left(s\mathcal{L}\{y\} - y(0)\right) + 4\mathcal{L}\{y\} = 0$$

$$(s^2 - 4s + 4)\mathcal{L}\{y\} = s\underbrace{y(0)}_{1} + \underbrace{y'(0)}_{1} - 4\underbrace{y(0)}_{1}$$

$$\mathcal{L}\{y\} = \frac{s-3}{s^2-4s+4}$$

$$= \frac{s-3}{(s-2)^2} = \frac{A}{s-2} + \frac{B}{(s-2)^2}$$

REPEATED LINEAR!

$$s - 3 = A(s-2) + B$$

$$\Rightarrow A = 1 \quad \& \quad -2A + B = -3$$

$$-2 + B = -3$$

$$B = -1$$

$$\mathcal{L}\{y\} = \frac{1}{s-2} + \frac{-1}{(s-2)^2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y = e^{2t} - te^{2t}}$$

"FORCING"

Ex] $y'' + 4y = 8, \; y(0) = 0, \; y'(0) = 3$

$8\mathcal{L}\{1\}$

$\underbrace{(s^2\mathcal{L}\{y\} - sy(0) - y'(0))} + 4\mathcal{L}\{y\} = \mathcal{L}\{8\}$

$(s^2+4)\mathcal{L}\{y\} - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{3} = \frac{8}{s}$

$(s^2+4)\mathcal{L}\{y\} = \frac{8}{s} + 3$

LEAVE SEPARATE!!

$\mathcal{L}\{y\} = \frac{8}{s(s^2+4)} + \frac{3}{s^2+4}$

$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$ ← IRREDUCIBLE QUADRATIC

$8 = A(s^2+4) + (Bs+c)s$

$A + B = 0, \; C = 0, \; 4A = 8$

$B = -2 \leftarrow A = 2$

$\mathcal{L}\{y\} = \frac{2}{s} + \frac{-2s}{s^2+4} + \frac{3}{s^2+4}$

WANT A 2 HERE!

$y(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$\boxed{y(t) = 2\cdot 1 - \underbrace{2\cos(2t) + \frac{3}{2}\sin(2t)}}$$

PARTICULAR SOL'N

HOMOGENEOUS SOL'N